The Math of Patches

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1 Introduction

Hearthstone is a popular card game played online. A recent card, "Patches" has been released that I wanted to look at for quantitative analysis. My work is shown below for inspection and revision.

In Hearthstone, each player has a deck of 30 cards. For the subsequent scenarios, we assume that one card in the deck is "Patches".

Patches has the subtext: "**Charge**: After you play a Pirate, summon this minion from your deck." As a result, the deck is conditionally thinned, that is, as long as Patches is not in the hand. Thus, it is useful to calculate the probability of drawing Patches. A Pirate is specific subset of cards that from this point on, we'll refer to as "**activators**" to highlight their use to activate Patches. In English, if the card Patches is not in your hand when you play an an activator, it'll be summoned from the deck into play. This is the optimal case that we aim to calculate for.

The beginning of a Hearthstone game starts with the mulligan phase. The mulligan is an opportunity for the player to replace unwanted cards and replace them with cards drawn from the deck. The mullliganned cards **cannot** be redrawn from the deck. The easiest way to think about where the mulliganned cards go is to put them into separate pile from deck, to be reintroduced into the deck again after the mulligan phase.

After the mulligan phase, the game is started and the mulliganned cards are reintroduced into the deck again. The cards are then reshuffled and cards are drawn normally from the top of the deck. We'll call this the game phase for purposes of distinguishing it from the mulligan phase

2 Math of the Mulligan Phase

Going first, the chances of getting Patches off the mulligan from a 3 card hand are:

Cards Mulliganed	Odds
1	3.7%
2	7.4%
3	11.1%

Table 1: *Mulligan chance of getting Patches versus cards mulliganned going first*

Calculating the chances of drawing exactly one card, Patches, off various mulligans can be calculated using this formula:

$$Probability\,of\,drawing\,Patches = 1 - The\,odds\,of\,not\,drawing\,Patches$$

Thus, the probability of drawing Patches can be calculated as follows:

$$Probability\,of\,drawing\,Patches = 1 - \left(\frac{n-x}{n} \times \frac{n-x-1}{n-1} \times ... \frac{n-x-d+1}{n-d+1}\right)$$

Where n is the number of cards in the deck, x is the number of cards of interest we wish to draw, and d is the number of cards drawn.

The astute observer will recognize that because the numbers cancels when the cards of interest is 1, the probability of drawing Patches can be simplified into $\frac{number\,of\,cards\,drawn}{size\,of\,deck}$. However, we'll calculate this for the simple case of 1 card mulliganned. Here, n, the deck size is 27, not 30 because we hold 2 cards in our hand and mulliganned one into a separate pile. Also, x is 1, the card Patches that we're interested in, and d is 1, because we draw 1 card.

$$Probability\,of\,drawing\,Patches\,for\,1\,card\,mulliganned = 1 - \frac{27-1}{27} = \frac{1}{27} = 3.7\%$$

We can apply the same equation to the player going second, who gets to mulligan 4 cards. Thus, going second, the chances of getting Patches off the mulligan from a 4 card hand are:

Cards Mulliganed	Odds
1	3.8%
2	7.7%
3	11.5%
4	15.4%

Table 2: *Mulligan chance of getting Patches versus cards mulliganned going second*

We'll calculate this again, this time for the case where 4 cards are mulliganned. Here, n, the deck size is 26, because we hold 0 cards in our hand and mulligan 4 into a separate pile. Also, x remains 1, Patches, and d is 4, because we draw 4 cards to replace the 4 we mulliganned.

$$Probability\,of\,drawing\,Patches\,for\,4\,cards\,mulliganned = 1 - \left(\frac{\cancel{25}}{26} \times \frac{\cancel{24}}{\cancel{25}} \times \frac{\cancel{23}}{\cancel{24}} \times \frac{22}{\cancel{23}}\right)$$

Thus, we can conclude that each additional card mulliganned early increases the odds of drawing Patches by approximately 4% with the player going first having a slightly lower probability. Later, we will show that increased activators in decks both increases the probability of the drawing an activator, while decreasing the odds of drawing Patches early which leads to higher incidence of optimal case play.

3 The Math of the Game Phase

For the second phase of the game, the first and second players start with a 27 card deck and 26 card deck respectively. Assuming that Patches has not been drawn after the mulligan phase, the probability of drawing Patches on the first turn is $\frac{1}{27}$ and $\frac{1}{26}$ for the first and second player respectively.

Drawing Patches from the mulligan and drawing Patches from the first turn are mutually exclusive events, meaning that one or the other can happen, but not both. Thus, we can just add the probabilities of either happening without having to worry about double counting. Thus, for the player who goes first who mulligans 1, the math looks like this:

$$Total\, probability = Probability\, of\, drawing\, on\, mulligan + Probability\, of\, drawing\, on\, first\, turn$$

Thus the combined odds for the player who goes first and mulligans 1 looks like this:

$$Combined\, probability = 0.037 + 0.037 = 7.4\%$$

These are the combined odds calculated for Patches going first:

Cards Mulliganed	Odds
1	7.4%
2	11.1%
3	14.5%

Table 3: *Total chance of getting Patches versus cards mulliganned going first. Notice that each value is simply Table 1's odds added to the probability of drawing Patches on the first turn.*

These are the combined odds calculated for Patches going second:

Cards Mulliganned	Odds
1	7.7%
2	11.5%
3	15.4%
4	19.2%

Table 4: *Mulligan chance of getting Patches versus cards mulliganned going second. Notice that each value is simply Table 2's odds added to the probability of drawing Patches on the first turn*

These odds and the odds of drawing an activator should be looked at in conjunction when building decks as to optimize win rate by increasing the chances of the optimal case occurring, that is for Patches to remain in the deck and for an activator to be in the hand by turn 1.

4 The Math of the "Pirates" (Activators)

From the introduction, we described the subset of the card pool "Pirates" as activators for the card Patches. We assume that all activators can be played turn 1 for cases of simplifying the mathematics here to make calculating the chances of drawing an activator more simple. While this doesn't exactly reflect conditions within the game, it is a close enough model that crucially helps us better predict the early part of a Hearthstone match.

When building a deck, a player has the option to choose how many activators he/she wishes to place into the deck. However, there are a limited amount of early game activators considered "playable", so we'll limit discussion from 2 to 6 activators per deck for calculations.

We can use the equation from earlier to calculate the probability of drawing an activator on the mulligan:

$$Probability\,of\,drawing\,any\,number\,of\,cards = 1 - \left(\frac{n-x}{n} \times \frac{n-x-1}{n-1} \times ... \frac{n-x-d+1}{n-d+1}\right)$$

Where n is the number of cards in the deck, x is the number of cards of interest we wish to draw, and d is the number of cards drawn.

We'll do an example of a player going first, with 6 activators in the deck, and 3 cards mulliganned:

The player going first has a 3 card hand. Regardless of how many cards the player mulligans, 3 will be removed from the deck. The only difference remains in the number of cards drawn after the mulligan. Thus calculation of the probability of drawing at least one activator is:

$$1 - probability\,of\,drawing\,no\,activators$$

Thus for 6 activators, we calculate the probability that we draw no activators at all:

$$Probability\,of\,drawing\,no\,activators\,for\,3\,cards\,mulliganned = \left(\frac{21}{27} \times \frac{20}{26} \times \frac{19}{25}\right)$$

Thus, the probability of drawing at least 1 activator for 6 activators and 3 cards mulliganned is:

$$1 - \left(\frac{21}{27} \times \frac{20}{26} \times \frac{19}{25}\right) = 54.5\%$$

The odds for drawing at least 1 activator going first and second are shown below assuming none of the cards held are already activators:

Number of Activators	Mulligan 1	Mulligan 2	Mulligan 3
2	7.4%	14.5%	21.3%
3	11.1%	21.4%	30.8%
4	14.8%	27.9%	39.5%
5	18.5%	34.2%	47.4%
6	22.2%	40.2%	54.5%

Table 5: *The chances of drawing at least one activator going first off the mulligan*

Number of Activators	Mulligan 1	Mulligan 2	Mulligan 3	Mulligan 4
2	7.7%	15.1%	22.2%	29.0%
3	11.5%	22.2%	31.9%	40.8%
4	15.4%	28.9%	40.8%	51.1%
5	19.2%	35.4%	48.8%	60.0%
6	23.0%	41.5%	56.2%	67.6%

Table 6: *The chances of drawing at least one activator going second off the mulligan*

The total probability of not drawing at least one activator on the first turn and mulligan is the multiplication of the probabilities of both occurring. The probability of not drawing on the first turn assumes that you don't draw an activator on the mulligan.

$$Total\, probability = Probability\, not\, drawing\, on\, mulligan \times Probability\, not\, drawing\, on\, first\, turn$$

Thus, the total probabilities of not drawing an activator off the mulligan or first turn are shown below for various mulligans:

Number of Activators	Mulligan 1	Mulligan 2	Mulligan 3
2	85.7%	79.2%	72.8%
3	79.0%	69.9%	61.5%
4	72.6%	61.4%	51.5%
5	66.4%	53.6%	42.9%
6	60.5%	46.5%	35.4%

Table 7: *The chances of not drawing at least one activator going first after the first turn*

Number of Activators	Mulligan 1	Mulligan 2	Mulligan 3
2	85.2%	78.4%	71.8%
3	78.3%	68.8%	60.2%
4	71.6%	60.2%	50.1%
5	65.2%	52.2%	41.4%
6	59.2%	45.0%	33.7%

Table 8: *The chances of not drawing at least one activator going second after the first turn*

5 Conclusion

Intuitively, increasing the number of activators and the number of cards mulliganned will increase the probability of drawing activators. However, doing so also increases the odds of drawing Patches, thus reducing the probability of the optimal case scenario. Unfortunately, we don't currently have the ability to calculate that probability, but simulating the scenario should allow for further calculations to be done if there exists further interest.

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